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The Rev. Charles Graves read the following note "on the Theory of Linear Differential Equations."

The equation

$$D^n y + A_1 D^{n-1} y + A_2 D^{n-2} y + \ldots\ldots + A_n y = X, \qquad (1)$$

in which A_1, $A_2, \ldots A_n$, and X, are any functions of x, and D stands for the symbol $\frac{d}{dx}$, may be brought, after n integrations, into the form

$$y + D^{-n} A_1 D^{n-1} y + D^{-n} A_2 D^{n-2} y + \ldots + D^{-n} A_n y = D^{-n} X$$
$$+ c_0 + c_1 x + \ldots c_{n-1} x^{n-1};$$

and this may be written as follows:

$$\phi(y) = D^{-n} X + c_0 + c_1 x + c_2 x^2 + \ldots c_{n-1} x^{n-1};$$

if we employ ϕ to denote the complex distributive operation

$$1 + D^{-n} A_1 D^{n-1} + D^{-n} A_2 D^{n-2} \ldots + D^{-n} A_n.$$

Operating now with the symbol ϕ^{-1} upon both sides of the last equation, we obtain the complete integral of the proposed one in the form

$$y = \phi^{-1}(D^{-n} X) + c_0 \phi^{-1}(1) + c_1 \phi^{-1}(x) + c_2 \phi^{-1}(x^2) \ldots + c_{n-1} \phi^{n-1}(x^{n-1}).$$

The term $\phi^{-1}(D^{-n} X)$ is evidently a particular integral of the proposed equation; whilst $\phi^{-1}(1)$, $\phi^{-1}(x) \ldots \phi^{-1}(x^{n-1})$ are particular integrals of the equation

$$D^n y + A_1 D^{n-1} y + A_2 D^{n-2} y + \ldots + A_n y = 0. \qquad (2)$$

This demonstration of the presence of n arbitrary constants in the complete integral, and of the mode of its composition, seems more simple and direct than those which are commonly given.

Putting U, u_0, u_1, $u_2 \ldots u_{n-1}$ in place of $\phi^{-1}(D^{-n} X)$, $\phi^{-1}(1)$, $\phi^{-1}(x)$, $\phi^{-1}(x^2) \ldots \phi^{-1}(x^{n-1})$, we may write

$$y - U = c_0 u_0 + c_1 u_1 + \ldots + c_{n-1} u_{n-1};$$

and differentiating this equation n times successively we have

$$
\begin{aligned}
Dy - DU &= c_0 Du_0 + c_1 Du_1 + \ldots + c_{n-1} Du_{n-1} \\
D^2y - D^2U &= c_0 D^2u_0 + c_1 D^2u_1 + \ldots + c_{n-1} D^2u_{n-1} \\
\ldots\ldots\ldots &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
D^ny - D^nU &= c_0 D^nu_0 + c_1 D^nu^1 + \ldots + c_{n-1} D^nu_{n-1}.
\end{aligned}
$$

The equation obtained by the elimination of the n constants $c_0, c_1, c_2, \ldots c_{n-1}$, from these last $n+1$ equations, being compared with the proposed equation (1), furnishes us with remarkable results.

The resulting equation* is

$$
\begin{aligned}
& S\,(\pm u_0 Du_1 D^2u_2 \ldots D^{n-1}u_{n-1} D^n y) \\
&= S\,(\pm u_0 Du_1 D^2u_2 \ldots D^{n-1}u_{n-1} D^n U),
\end{aligned}
$$

which, being arranged according to the differential coefficients of y, becomes

$$
\begin{aligned}
& S\,(\pm u_0 Du_1 D^2u_2 \ldots D^{n-2}u_{n-2} D^{n-1}u_{n-1})\, D^n y \\
& - S\,(\pm u_0 Du_1 D^2u_2 \ldots D^{n-2}u_{n-2} D^n u_{n-1})\, D^{n-1} y + \ldots \\
& \ldots \pm S\,(\pm u_0 D^2u_1 D^3u_2 \ldots D^{n-1}u_{n-2} D^n u_{n-1})\, Dy \\
& \mp S\,(\pm Du_0 D^2u_1 \ldots D^{n-1}u_{n-2} D^n u_{n-1})\, y \\
& = S\,(\pm u_0 Du_1 D^2u_2 \ldots D^{n-1}u_{n-1} D^n U).
\end{aligned}
$$

Putting this expression, for the sake of brevity, into the form

$$S_n D^n y - S_{n-1} D^{n-1} y + \ldots \pm S_1 Dy \mp S_0 y = S', \}$$

we have the following relations:

$$- S_{n-1} = A_1 . S_n \qquad (3)$$

$$S_{n-2} = A_2 . S_n \qquad (4)$$

$$\ldots\ldots = \ldots\ldots$$

---

* $S\,(\pm u_0 Du_1 \ldots D^{n-1}u_{n-1} D^n y)$ is here used to denote the sum of all the terms derived from $u_0 Du_1 \ldots D^{n-1}u_{n-1} D^n y$ by the permutation of the elements $u_0, u_1 \ldots u_{n-1}, y$; each term being regarded as positive or negative according as it may be deduced from that first term by means of an odd or even number of interchanges of two letters.

$$\pm S_1 = A_{n-1}.S_n \qquad (5)$$

$$\mp S_0 = A_n.S_n \qquad (6)$$

$$S' = X.S_n,$$

the last of which shows that U, the particular integral of equation (1), is determined if u_0, u_1, $u_2 \dots u_{n-1}$, the particular integrals of (2), be known. The remaining equations indicate the relations which exist between $A_1, A_2, \dots A_n$, and u_0, u_1, $\dots u_{n-1}$. We are not able to derive the integrals $u_0, u_1, \dots u_{n-1}$ from the equations just given, any more than we can determine the roots of an algebraic equation from the well known relations between them and its coefficients. In fact, if we were to multiply the equations (3), (4), (5), (6) by $D^{n-1}u_0 D^{n-2}u_0 \dots Du_0$, u_0, and add to their sum the identical equation,

$$S_n D^n u_0 = S_n D^n u_0,$$

we should eliminate the other roots $u_1\, u_2 \dots u_{n-1}$, but at the same time reproduce the original differential equation.

All the preceding reasoning applies whenever D denotes, not merely the operation of taking the differential coefficient, but *any* distributive operation such that

$$D^n\,(c_0 + c_1 x + \dots + c_{n-1}x^{n-1}) = 0.$$

The results obtained hold good, therefore, in the case of equations in finite differences.

As regards the case of differential equations, it is worthy of notice that the equation (3) admits of integration independently of any relation between the functions $u_0, u_1, \dots u_{n-1}$.

Since

$$S_{n-1} = D\,S_n,$$

we have

$$S\,(\pm\, u_0 D u_1 D^2 u_2 \dots D^{n-2}u_{n-2} D^{n-1}u_{n-1}) = e^{-\int A_1 dx.}$$

And it follows from this that the left hand member of equation (1) becomes a complete determinant when multiplied by $e^{-\int A_1 dx}$.

The problem of expressing the coefficients of the differential equation (2), in terms of its particular integrals, has been treated by M. G. Libri, in a very elegant memoir on Linear Differential Equations, printed in the tenth volume of Crelle's Journal. He has given the following formula to determine A_1:

$$A_1 = -\frac{nDu_0}{u_0} - \frac{(n-1)\,D^2\left(\dfrac{u_1}{u_0}\right)}{D\left(\dfrac{u_1}{u_0}\right)} - \frac{(n-2)\,D^2\left(\dfrac{D\left(\dfrac{u^2}{u_0}\right)}{D\left(\dfrac{u_1}{u_0}\right)}\right)}{D\left(\dfrac{D\left(\dfrac{u^2}{u_0}\right)}{D\left(\dfrac{u_1}{u_0}\right)}\right)} - \&c.,$$

and merely indicated the method of obtaining expressions for the other coefficients. From the nature of this method, however, it is easy to see that it would be scarcely possible to write down the values of the higher coefficients, in terms of u_0, u_1, &c., on account of their extreme complexity. M. Libri has noticed that the expression given above for A_1 is, from the nature of the case, a symmetrical function of u_0, u_1, &c.; though this is not indicated by its actual form. To exhibit it as a symmetrical function of those particular integrals we must execute in it all their possible permutations, and then take the sum of the results. This operation considerably increases the complexity of the formula.

[In the notice of Mr. Donovan's Lamp (p. 75), it was omitted to be stated that it burned with a brilliant light during the sitting of the Academy.]